Physics 361 **Final exam** **December 11, 2002**

Due at 3:30. No book or notes. Brief answers are in [[...]] .

It may be useful to recall that $\int dx\ f(x)\delta(u_0 - u(x)) = [f(x(u))\ |dx/du|]_{u=u_0}$.

F.1 Comparing two metals Two free-electron-like metals, A, and B both have a single valence electron, but the metal B has three times as many electrons per unit volume as metal A. Briefly justify your answers:

a) What is the electric conductivity of metal B relative to A, assuming the scattering time τ is the same for both? [[3]]
b) What is the electronic specific heat of B relative to A? [[$3^{1/3}$]]
c) What is the thermopower of B relative to A, *i.e.,* the electric field induced by a thermal gradient? [[$3^{-2/3}$]]
d) What is the Hall coefficient of B relative to A, *i.e.,* the transverse electric induced by a magnetic field and a longitudinal current? [[1/3]]

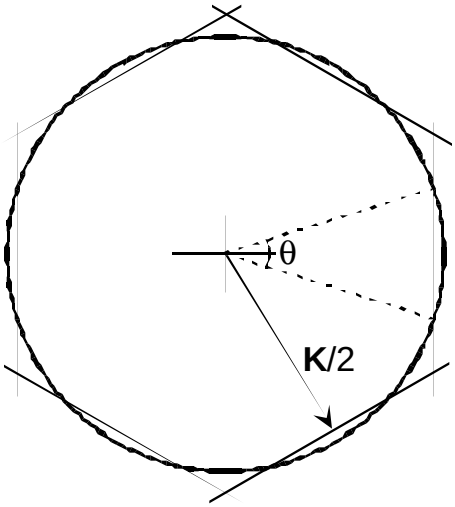
F.2 triangular semiconductor Electrons in a certain tight-binding band of a triangular lattice have

$$\mathcal{E}(\mathbf{k}) = C - \gamma \left[\cos(a\mathbf{k}\cdot\hat{\mathbf{x}}) + \cos(a\mathbf{k}\cdot(\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}})) + \cos(a\mathbf{k}\cdot(-\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}))\right],$$

where a is the distance between nearest neighbors in the xy plane . This band has a miniminum at 0 and a maximum at the zone corners, such as $\mathbf{k}_0 = \frac{4\pi}{3a}\hat{\mathbf{x}}$.

a) If the distance a is increased greatly, does the coefficient γ stay about the same, become much larger or become much smaller? Justify your answer briefly. [[becomes much smaller. The $\mathbf{k}$ dependence of $\mathcal{E}$ arises because of the interaction of the atomic wavefunctions. If the atoms are far apart, the $\mathbf{k}$ dependences goes to zero; hence γ goes to zero.]]
b) What is the effective mass m^* near $\mathbf{k} = 0$ in terms of γ and a ? [[$\hbar^2/(a^2\gamma)$]]
c) Electrons are added to the empty band so that there are .01 electrons per unit cell, to form semiconductor A. Electrons are removed from the full band so that it contains only 1.99 electrons, forming semiconductor B. Find the specific heat of B relative to that of A at a low temperature. [[B has .01 holes per unit cell. To find their specific heat, one must find the density of states at their fermi surface near the top of the band. Near this maximum, the $\mathcal{E}(k)$ has a quadratic form. Differentiating as in b) one finds that the mass is equal to that at $\mathbf{k} = 0$. Since the density and the $\mathcal{E}$ are the same for the holes in B and the electrons in A, they have the same density of states at the Fermi surface and hence the same specific heat.]]

F.3 Hexagonal orbits



A two-dimensional free-electron-like metal has a triangular lattice structure and two electrons per unit cell. Thus there are two partially filled bands. The second band is nearly empty with the occupied states forming three lens-shaped regions in the reciprocal lattice. In homework, you deduced that each lens subtends an angle θ of about 35 degrees or .62 radians.

a) How many electrons per unit cell occupy the second band? [[$6(\theta - \sin\theta)/\pi \simeq .07$]]
b) How many electrons per unit cell occupy the lowest band? [[2− a)]]

If a uniform magnetic field H is applied perpendicular to the plane of the metal, electron states change their Bloch wavevector $\mathbf{k}$ with time in a periodic way. The orbital period $\mathcal{T}$ is given by $\mathcal{T} = \frac{2\pi m^* c}{eH}$, where m^* is the cyclotron effective mass.

c) Find the cyclotron effective mass m^* for the Fermi surface of the second band. In other words, find $\mathcal{T}$ relative to the $\mathcal{T}$ in the absense of a lattice. [[$\mathcal{T} \int dk/|\mathbf{v}(\mathbf{k})|$ integrated around the orbit enclosing the occupied states. This $\mathbf{v}(\mathbf{k}) = \partial\mathcal{E}/\partial\mathbf{k}$. Since the bands are nearly free electronlike, their speed is the free-electron speed v_0. The period $\mathcal{T}$ is just that for the free electons times the ratio of second-band perimeter to the free electron Fermi sphere circumference: *viz.* $2\theta k_F/2\pi k_F$. $\mathcal{T}/\mathcal{T}_0 = m^*/m_0 = \theta/\pi \simeq .2$]]
d) The energy $\mathcal{E}_1$ is shifted downward slightly by the lattice potential. At the center of the zone face, where $\mathbf{k} = \mathbf{K}/2$, $\mathcal{E}_1(\mathbf{K}/2) = \mathcal{E}_0(\mathbf{K}/2) - \Delta$, where $\mathcal{E}_0(\mathbf{k}) \equiv \hbar^2\mathbf{k}^2/(2m)$ is the free-electron energy. Similarly, at the zone corner, $\mathcal{E}_1(\mathbf{k}) = \mathcal{E}_0 - \alpha\Delta$, where α is a numerical constant. Within the nearly-free-electron approximation, how large should Δ become relative to $\mathcal{E}_0(\mathbf{K}/2)$ in order to turn the metal into an insulator? [[In order to have all the electrons in the lowest band, the top of the lowest band (at the zone corner) must be lower than the bottom of the second band (at $\mathbf{k}/2$). $\Delta \geq \mathcal{E}_0(\mathbf{K}/2)/(3(1+\alpha))$]]

F.4 Graphite conductivity Graphite has a band whose energy is well approximated by

$$\mathcal{E}(\mathbf{k}) = \frac{\hbar^2}{2m_\perp}(k_x^2 + k_y^2) + \frac{\hbar^2}{2m_z}k_z^2$$

where $m_z \gg m_\perp$. The fermi surface, with energy $\mathcal{E}_F$, is thus a cigar-shaped ellipsoid

pointed in the z direction. We denote the $\mathbf{k}$ at the tip of this ellipsoid as $\mathbf{k}_p$. The group velocity of state a state with wavevector $\mathbf{k}$ is denoted by $\mathbf{v}(\mathbf{k})$.

a) What is $\mathbf{v}(\mathbf{k})$ for $\mathbf{k}$ on the Fermi surface in the x-y plane, as a function of $|\mathbf{v}(\mathbf{k}_p)|$? Tell the direction as well as the magnitude. [[$|v_k| = (m_z/m_\perp)^{1/2}|v(\mathbf{k}_p)$ the direction is along $\mathbf{k}$, in the x-y plane.]]

According to Chapter 13, the conductivity tensor σ_{ij} defined by $j_i = \sigma_{ij}E_j$, is given by

$$\sigma_{ij} = \text{constant} \int d^3k \ \tau \ v_i(\mathbf{k})v_j(\mathbf{k})\frac{\partial f}{\partial \mathcal{E}}$$

where τ is the relaxation time and $f(\mathcal{E})$ is the probability that a state with energy $\mathcal{E}(\mathbf{k})$ is occupied in equilibrium.

b) Find σ_{zz}/σ_{xx} in the low-temperature limit. [[$m_\perp/m_z$]]

An electric field $\mathbf{E}$ is applied at a 45 degree angle to the z axis, so that $\mathbf{E} = E(\hat{\mathbf{x}} + \hat{\mathbf{z}})$.

c) Find the direction of the electron current $\mathbf{j}$. [[$j_z/j_x = \sigma_{zz}/\sigma_{xx} = m_\perp/m_z$ Current is nearly in the x-y plane and makes an angle $m_\perp/m_z$ with it.]]

It may be useful to recall that $\int dx \ f(x)\delta(u_0 - u(x)) = [f(x(u))dx/du]_{u=u_0}$.