Physics 361 Solutions to Problem Set 5

November 26, 2002

1 Surface Plasmon and Thin Film Plasmon

The surface plasmon dispersion relation is

$$q^2 = \left(\frac{\omega}{c}\right) \frac{\omega_p^2 - \omega^2}{\omega_p^2 - 2\omega^2}$$

This expression diverges at $\omega = \omega_p/\sqrt{2}$, [[indicating that this is maximum the highest frequency obtainable for this mode as $q^2 \to \infty$. The bulk plasmon mode has a different field configuration and oscillates at the higher frequency ω_p.]] For the long wavelength limit, [[$\omega \to 0$ or $\omega \to \omega_p$. The latter possibility is ruled out be the argument above. Thus]] $\omega(q) \ll \omega_p$, leading to $\omega(q) \approx c|q|$. A plot of $\omega(|q|)$ appears at the end of these solutions.

Thin film plasmon Let n_0 be the uniform number density of the unperturbed 2D electron gas. If electrons are displaced by $\mathbf{u}(x, y) = \hat{\mathbf{x}} u_0 \sin qx$, the density profile becomes

$$\begin{aligned} n_s(x) &= n_0 \int dx' \, \delta[x - (x' + u_0 \sin qx')] \\ &= n_0 / |\partial_{x'}(x' + u_0 \sin qx')|_{x' + u_0 \sin qx' = x} \\ &\approx n_0 / |1 + q u_0 \cos qx| \\ &\approx n_0 (1 - q u_0 \cos qx) \end{aligned}$$

In the second line, we have used standard rules for evaluating integrals of delta functions. We assume $qu_0 \ll 1$, so that the delta function is non-zero at $x' = x$, to leading order in qu_0. After accounting for the uniform background charge of the ions, the charge density profile becomes $\rho(x) = e n_0 q u_0 \cos qx$.

For a line charge with density ρ, the field can be evaluated by Gauss' law using a cylinder of length L:

$$\begin{aligned} 2\pi r_\perp L |E| &= 4\pi \rho L \\ |E| &= 2\rho / r_\perp \end{aligned}$$

The resulting electrostatic potential is $-2\rho \ln(r_\perp / r_0)$, where r_0 is an arbitrarily chosen radius where the potential vanishes.

[[The potential is given by

$$\phi(x) = -\int dx'\, 2\rho(x') \ln\left(\frac{|x-x'|}{r_0}\right)$$

where $\rho(x) = e\ n_0 q\ u_0 \cos qx$. To avoid the unphysical divergence associated with the log, we integrate by parts noting that $\rho(x) = \frac{d}{dx}\ e\ n_0 u_0 \sin(qx)$. The potential is given by

$$\begin{aligned}\phi(x) &= -\int dx'\, 2e\ n_0 u_0 \sin(qx') \frac{1}{|x-x'|} \\ &= 2\pi e n_0 u_0 \cos qx.\end{aligned}$$

]] With charge density $\rho(x) = e n_0 q u_0 \cos qx$, the potential energy per unit area becomes $U = \rho\phi = 2\pi q u_0^2 e^2 n_0^2 \cos^2 qx$ which has the spatial average $\pi q u_0^2 e^2 n_0^2$. If the charge displacement oscillates with frequency ω, the kinetic energy per unit area is

$$K = \frac{1}{2} m_e n_0 (\partial_t \mathbf{u})^2 = \frac{1}{2} m_e n_0 u_0^2 \omega^2 \sin^2 qx$$

Spatial averaging introduces a factor of $1/2$, so that the effective mass per unit area is $\frac{1}{2} m_e n_0$. The effective spring constant (relative to u_0) is $2\pi q e^2 n_0^2$, leading to the oscillation frequency

$$\omega = (4\pi e^2 n_0 / m_e)^{1/2} q^{1/2}$$

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The thin film plasmon group velocity goes as $q^{-1/2}$, which diverges in the long wavelength limit. [[This unphysical divergence occurs because we neglected retardation effects in our electrostatic formula for $\phi(x)$.]]

more detail on handling the divergent potential The potential from the 2D electron gas is

$$\begin{aligned}\phi(x,z) &= -\int dx'\, 2\rho(x') \ln\left(\frac{\sqrt{(x-x')^2+z^2}}{r_0}\right) \\ &= -\int dx'\, 2\rho(x'+x) \ln\left(\frac{\sqrt{x'^2+z^2}}{r_0}\right) \\ \rho(x'+x) &= e n_0 q u_0 (\cos qx' \cos qx - \sin qx' \sin qx) \\ \phi(x,z) &= -2e n_0 q u_0 \cos qx \int dx'\ \cos qx' \ln\left(\frac{\sqrt{x'^2+z^2}}{r_0}\right) \\ &= -2e n_0 u_0 \cos qx \cdot I(qz)\end{aligned}$$

where

$$I(s) = \int_{-\infty}^{+\infty} dt\ \cos t \cdot \ln\left(\frac{\sqrt{s^2+t^2}}{q r_0}\right)$$

Note that the $\sin qx'$ term in the charge density is odd and makes no contribution to the integral. We are interested in the potential at the surface and therefore want $I(0)$. This integral is formally divergent, but there is a physically relevant regularization, in which the contribution from nearby charges is "more important" than the contribution from distant charges. A similar situation arises in the calculation of the Madelung constant for ionization from a lattice.

$$\begin{aligned} I(0) &\equiv \lim_{\eta\to 0^+} 2\int_0^\infty dt\, e^{-\eta t}\cos t(\ln t - \ln qr_0) \\ &= \lim_{\eta\to 0^+} 2\mathrm{Re}\int_0^\infty dt\, e^{(i-\eta)t}(\ln t - \ln qr_0) \\ &= \lim_{\eta\to 0^+} 2\mathrm{Re}\int_0^\infty dt\, e^{(i-\eta)t}\ln t \\ &= \lim_{\eta\to 0^+} 2\mathrm{Re}\int_0^\infty i\,dt\, e^{-(1+i\eta)t}(\ln t + i\pi/2) \\ &= -\pi \end{aligned}$$

In the last step, the contour of integration has been rotated to the positive imaginary axis.

The electrostatic potential at the surface is therefore $\phi(x, z=0) = 2\pi e n_0 u_0 \cos qx$.

2 Phonon and electron specific heats

The phonon specific heat per unit volume for a lattice with sound speed c is, in the low temperature limit,

$$c_v^{ph} = \frac{2}{5}\pi^2 k_B \left(\frac{k_B T}{\hbar c}\right)^3$$

The electronic specific heat per unit volume is $c_v^{el} = \gamma T$, with

$$\gamma = k_B^2 \frac{\pi^2}{3} g(\mathcal{E}_F)$$

where $g(\mathcal{E}_F) = 3n/(2\mathcal{E}_F)$ is the density of states at the fermi energy. The two specific heats are equal at $T_\times$, giving

$$\left(\frac{k_B T_\times}{\mathcal{E}_F}\right)^2 = \frac{5}{4} n \left(\frac{\hbar c}{\mathcal{E}_F}\right)^3$$

For a 3D Fermi gas, $n = (mv_F/\hbar)^3/(3\pi^2)$ and $\mathcal{E}_F = \frac{1}{2}mv_F^2$, so that $n/\mathcal{E}_F^3 = 8/(3\pi^2(v_F/\hbar)^3)$.

$$\frac{k_B T_\times}{\mathcal{E}_F} = \sqrt{\frac{10}{3\pi^2}\left(\frac{c}{v_F}\right)^3}$$

In terms of the Debye temperature, the phonon molar specific heat is

$$c_v^{ph} = \frac{12}{5}\pi^4 n_i k_B \left(\frac{T}{\Theta_D}\right)^3$$

where $n_i = 6.022 \times 10^{23}$, the number lattice ions per mole. On, the other hand, the electronic molar specific heat is $c_v^{el} = \gamma T$. The crossing temperature in terms of Θ_D and γ is

$$T_\times^2 = \frac{5}{12\pi^4} \frac{\gamma}{n_i k_B} \Theta_D^3$$

The Debye temperatures for Na and Cu (150 K and 315 K, respectively) can be found on the periodic table inside the front cover of A & M, and γ values appear on page 48 (3.5 and 1.6 $\times 10^{-4}$ cal mol^{-1} K^{-2}, respectively.) For Na $T_\times = 1.6$ K and for Cu $T_\times = 3.3K$.

3 Friedel oscillations

For the 1-D case the wavefunction for a state with momentum k is $\psi_k(x) = A \sin kx$. The density at zero temperature is a sum of state below the fermi energy:

$$\begin{aligned} n(x) &= \int_{-k_F}^{k_F} dk \, |\psi_k(x)|^2 \\ &= A^2 k_F \left(1 - \frac{\sin 2k_F x}{2k_F x}\right) \\ &= n(\infty) \left(1 - \frac{\sin 2k_F x}{2k_F x}\right) \end{aligned}$$

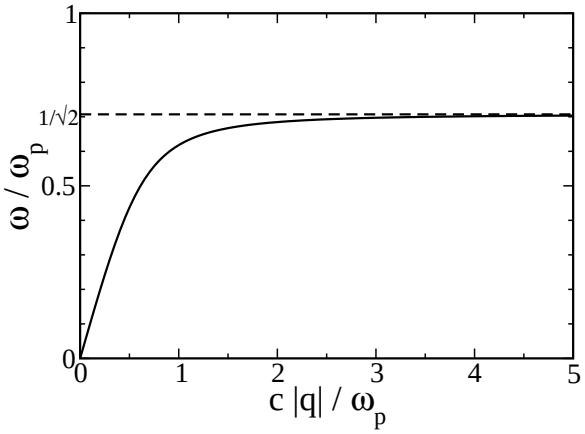
In the 3-D case, $\psi_{\mathbf{k}}(\mathbf{x}) = A \sin k_x x e^{i(k_y y + k_z z)}$, $|\psi_{\mathbf{k}}(\mathbf{x})|^2 = A^2 \sin^2 k_x x$, but now the density integral is over a spherical Fermi sea:

$$\begin{aligned} n(x) &= \int_{|\mathbf{k}<k_F|} d^3k |\psi_{\mathbf{k}}(\mathbf{x})|^2 \\ &= \int_{-k_F}^{k_F} dk_x \, A^2 \sin^2 k_x x \cdot \pi(k_F^2 - k_x^2) \end{aligned}$$

where the k_y and k_z integrals have been performed at fixed k_x. The result of the k_x integral is

$$n(x) = n(\infty) \left(\frac{\sin 2k_F x}{8(k_F x)^3} - \frac{\cos 2k_F x}{(k_F x)^2} \right) \tag{1}$$

Plasmon dispersion



Friedel oscillations

